Measurement Contextuality and Planck's Constant (arXiv:1711.08066)

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Abstract

We show that ***contextual measurements*** in finite-dimensional systems have formulations within the Wigner-Weyl-Moyal (WWM) formalism that require ***higher than order*** $\hbar^0$ terms to be included in order to violate the classical bounds on their expectation values.

As a result, we show that measurement contextuality as a resource is equivalent to orders of $\hbar$ ***as a resource.***

Introduction

Contextuality is a quantum resource whose presence in an experiment has been shown to be equivalent to ***negativity*** in the associated discrete Wigner functions and Weyl symbols of the states, operations, and measurements that are involved.

Here we complete the characterization of the relationship between orders of $\hbar$ in WWM and contextuality by showing that ***measurement contextuality*** is also equivalent to non-classicality as dictated by powers of $\hbar$.

We find:

- contextuality requires higher than order $\hbar^0$ terms in the $\hbar$ expansion of observables within the WWM formalism to obtain expectation values that violate classical bounds
- qubits exhibit *state-independent contextuality* while odd-dimensional qudits also exhibit *state-dependent contextuality*

Background

Definition: Context of a Measurement

Consider a ***projection*** of a quantum state onto a ***rank*** $n \geq 2$ ***subspace*** of its Hilbert space

- can be decomposed into a sum of smaller rank projectors in *many ways*
- fixing a subset of the terms in a sum of such projectors, there are many choices for the remaining terms

Each non-commuting decomposition of the remaining terms corresponds to a "context" of a measurement.

Instead of projectors we may speak instead about observables.

- rank $n \geq 2$ subspace is then a degenerate eigenspace of some observable
- ***different contexts*** correspond to ***different choices of complete sets of commuting observables***
  - eigenstates are the projectors onto the different contexts

For instance, consider two qubits:

- measurement of $\hat{X}\hat{I}$
  - corresponds to a projection onto a subspace of rank two
  - can be performed in the context of:
    - $\{\hat{X}\hat{I}, \hat{I}\hat{X}\}$ or $\{\hat{X}\hat{I}, \hat{I}\hat{Z}\}$
  - the operators in each set commute with each other
  - however, the two operators that distinguish these contexts, $\hat{I}\hat{X}$ and $\hat{I}\hat{Z}$ anticommute
  - the product of the operators in each set anticommute with each other
- hence the outcome of a measurement of $\hat{X}\hat{I}$ is *dependent* on the choice of context
  - each set corresponds to a projection onto the full rank four Hilbert space and is a ***separate context*** for $\hat{X}\hat{I}$

These two sets correspond to the first row and column of the Peres-Mermin square shown in Table 1 (the third element in the row and/or column is redundant—its outcome is determined by the first two measurements)

**This work was supported by AFOSR award no. FA9550-12-1-0046.*

WWM Formalism Crash Course

Odd-Dimensional Qudits

Setting $\hbar = d/2\pi$, and letting $(\boldsymbol{\lambda}_p, \boldsymbol{\lambda}_q)$ and $(\boldsymbol{x}_p, \boldsymbol{x}_q)$ be in $(\mathbb{Z}/d\mathbb{Z})^{2n}$, we define the translation operator

$$\hat{T}(\boldsymbol{\lambda}_p, \boldsymbol{\lambda}_q) = \omega^{-\boldsymbol{\lambda}_p\cdot\boldsymbol{\lambda}_q(d+1)/2}\hat{Z}^{\boldsymbol{\lambda}_p}\hat{X}^{\boldsymbol{\lambda}_q}.$$

where $\omega \equiv \exp 2\pi i/d$ and $(d+1)/2$ is equivalent to $1/2$ in mod odd-d arithmetic. The reflection operator is defined as

$$\hat{R}(\boldsymbol{x}_p, \boldsymbol{x}_q) = d^{-n} \sum_{\substack{\boldsymbol{\xi}_p, \boldsymbol{\xi}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^n}} e^{\frac{2\pi i}{d}(\boldsymbol{\xi}_p, \boldsymbol{\xi}_q)\mathcal{J}(\boldsymbol{x}_p, \boldsymbol{x}_q)^T} \hat{T}(\boldsymbol{\xi}_p, \boldsymbol{\xi}_q).$$

The Weyl symbol of operator $\hat{\rho}$ can be expressed as the coefficient of the operator expanded in the $\hat{R}$ basis:

$$\hat{\rho} = \sum_{\substack{\boldsymbol{\xi}_p, \boldsymbol{\xi}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^n}} \hat{R}^\dagger(\boldsymbol{x}_p, \boldsymbol{x}_q) W_{\hat{\rho}}(\boldsymbol{x}_p, \boldsymbol{x}_q).$$

If $\hat{\rho}$ is a state, $W_{\hat{\rho}}(\boldsymbol{x})$ is the corresponding Wigner function.

While the WWM formalism for odd-dimensional qudits can be made with the two generators, $\hat{p}$ and $\hat{q}$, the WWM formalism for qubits requires three generators.

Qubits

Let ξ_p, ξ_q and ξ_r be three real generators of a Grassmann algebra $\mathcal{G}_3$. Hence,

$$\xi_j\xi_k + \xi_k\xi_j \equiv \{\xi_j, \xi_k\} = 0, \quad \text{for } j,k \in \{1,2,3\},$$

where we can identify $\xi_p \equiv \xi_1$, $\xi_q \equiv \xi_2$ and $\xi_r \equiv \xi_3$.
To quantize our algebra, the three generators ξ_k become the Pauli operators $\hat{\xi}_k$.
It can be shown that the operator

$$\hat{T}(\rho) = \exp\left(\frac{2i}{\hbar}\sum_k \hat{\xi}_k\rho_k\right)$$

corresponds to a translation operator and the dual to the translation operator $\hat{T}$ is

$$\hat{R}(\xi) = \int \exp\left(-\frac{2i}{\hbar}\sum_k \xi_k\rho'_k\right)\hat{T}(\rho')\mathrm{d}^3\rho',$$

which corresponds to a reflection operator.
These $\hat{R}$ serve as a complete operator basis for any Hilbert space operator $\hat{g}$ under Grassmann integration:

$$\hat{g} = \int \hat{R}(\xi)g(\xi)\mathrm{d}^3\xi.$$

Groenewold's Rule:

$$W_{AB}(\boldsymbol{x}) = W_A(\boldsymbol{x})W_B(\boldsymbol{x}) + \mathcal{O}(\hbar),$$

for odd d,

$$W_{AB}(\xi) = W_A(\xi)W_B(\xi) + \mathcal{O}(\hbar),$$

for $d = 2$.

Theorem: Measurement Contextuality

A pure state $\hat{\rho} \equiv |\Psi\rangle\langle\Psi|$ exhibits measurement contextuality under measurement by some observable $\hat{\Sigma}$ under contexts $\hat{\Sigma}\hat{\Sigma}_k$ if the Wigner function of the operators, $W_{\Sigma\Sigma_k}$, must be treated at an order higher than $\hbar^0$ to compute the expectation values:

$$\langle\Psi|\hat{\Sigma}\hat{\Sigma}_k|\Psi\rangle = \begin{cases} \int_{-\infty}^{\infty} W_{\Sigma\Sigma_k}(\xi)\tilde{W}_\rho(\xi)\mathrm{d}\xi. & \text{for } d = 2, \\ \sum_{\boldsymbol{x}} W_{\Sigma\Sigma_k}(\boldsymbol{x})W_\rho(\boldsymbol{x}) & \text{for odd } d. \end{cases}$$

Examples:

Peres-Mermin Square (Qubit State-Independent Contextuality)

In Table 1 below, multiplying together any of the operators in a row or column corresponding to a context requires multiplying two Pauli operators $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in each qubit tensor factor:

	Meas. # 1	Meas. # 2	Meas. # 3	Outcome
Meas. # 1	$\hat{\sigma}_{p_1}$	$\hat{\sigma}_{p_2}$	$\hat{\sigma}_{p_1}\hat{\sigma}_{p_2}$	$+1$
Meas. # 2	$\hat{\sigma}_{r_2}$	$\hat{\sigma}_{r_1}$	$\hat{\sigma}_{r_1}\hat{\sigma}_{r_2}$	$+1$
Meas. # 3	$\hat{\sigma}_{p_1}\hat{\sigma}_{r_2}$	$\hat{\sigma}_{r_1}\hat{\sigma}_{p_2}$	$\hat{\sigma}_{q_1}\hat{\sigma}_{q_2}$	$+1$
Outcome	$+1$	$+1$	-1	$+1$ / -1

Table 1: The Peres-Mermin Square. Every observable commutes with every other observable in its row and column, but anticommutes with the other four observables. Taking the measurements row-wise produces only $+1$ outcomes, while the measurements column-wise produce two $+1$ outcomes and a -1 outcome, the product of which is -1 as shown in the bottom-rightmost cell. Hence, the context of the measurement scheme determines the outcomes.

We can use Groenewold's Rule directly and find:

$$\begin{aligned} W_{\hat{\sigma}_a\hat{\sigma}_b}(\xi) &= (\alpha_1 i\xi_r\xi_q + \beta_1 i\xi_p\xi_q + \gamma_1 i\xi_p\xi_r) \\ &\quad \times (\alpha_2 i\xi_r\xi_q + \beta_2 i\xi_p\xi_q + \gamma_2 i\xi_p\xi_r) + \mathcal{O}(\hbar) \\ &= 0 + \mathcal{O}(\hbar). \end{aligned}$$

Therefore, from the Theorem, Pauli qubit operators are contextual for ***all states***, as they all require the order $\hbar^1$ term of the measurement operator.

KCSB Construction (Odd d Qudit State-Dependent Contextuality)

Consider the set of projectors

$$\Gamma^2 = \{\hat{\Pi}_1\hat{\Pi}_3, \hat{\Pi}_1\hat{\Pi}_4, \hat{\Pi}_2\hat{\Pi}_4, \hat{\Pi}_2\hat{\Pi}_5, \hat{\Pi}_3\hat{\Pi}_5, \hat{\Pi}_3\hat{\Pi}_1, \hat{\Pi}_4\hat{\Pi}_1, \hat{\Pi}_4\hat{\Pi}_2, \hat{\Pi}_5\hat{\Pi}_2, \hat{\Pi}_5\hat{\Pi}_3\},$$

which define the observable

$$\hat{\Sigma}_{\Gamma^2} = \sum_{\hat{\Pi}_i\hat{\Pi}_j \in \Gamma^2} \hat{\Pi}_i\hat{\Pi}_j,$$

where the projectors $\hat{\Pi}_i$ are defined as in Figure 1:

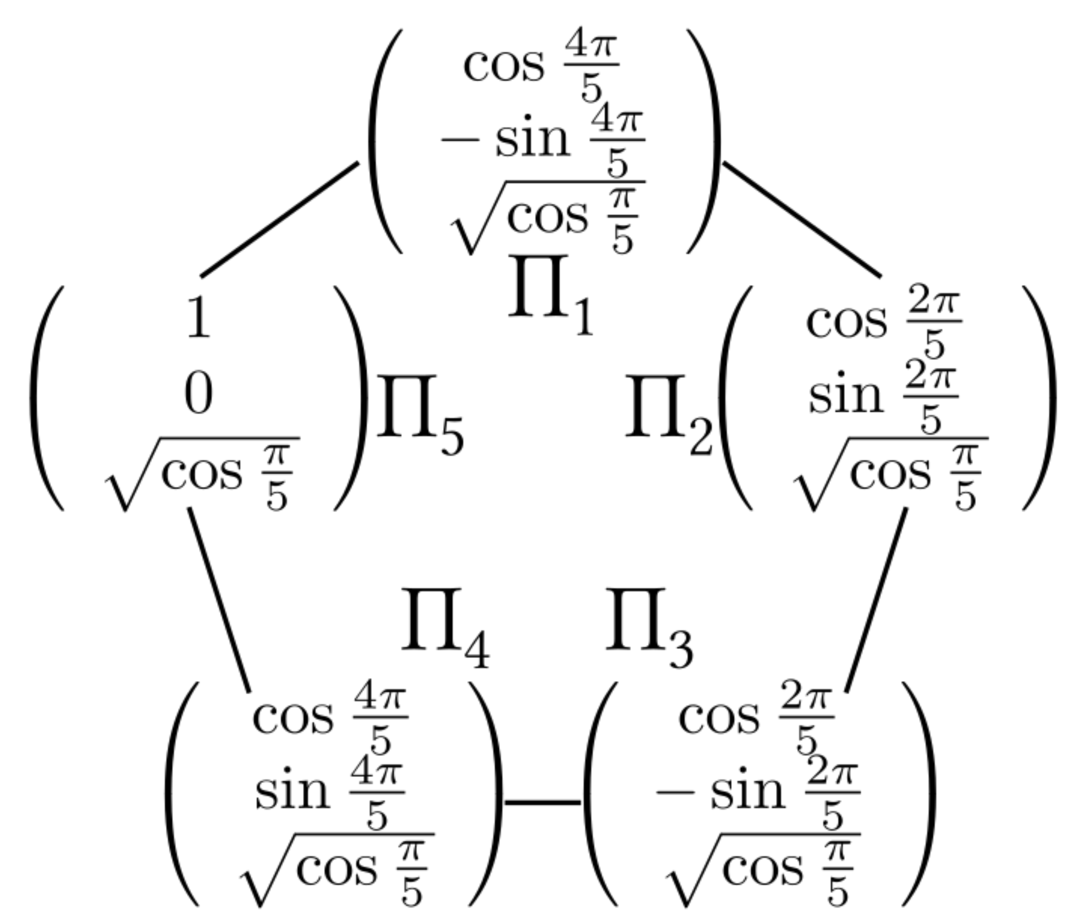

Figure 1: The KCSB contextuality construction for a qutrit. The five Π_i projectors are outer products of the vectors above (after normalization) and commute with each other if they share an edge.

$$\implies \hat{\Pi}_i\hat{\Pi}_{i\oplus 1} = 0.$$

Any classical outcome $\{0, +1\}$ one preassigns to the measurement must obey the same relationship. Adjacent vertices therefore cannot both be assigned the outcome $+1$, and so

$$\langle\Psi|\hat{\Sigma}_{\Gamma^2}|\Psi\rangle_{\text{CM}} \leq 2.$$

This upper bound is higher if the above expectation value is evaluated quantum mechanically. Namely, an eigenstate ϕ_3 of $\hat{\Sigma}_\Gamma$ can be shown to saturate the quantum bound:

$$\langle\Psi|\hat{\Sigma}_{\Gamma^2}|\Psi\rangle_{\text{QM}} \leq 5 - \sqrt{5} \approx 2.76393 = \langle\phi_3|\hat{\Sigma}_{\Gamma^2}|\phi_3\rangle.$$

$\hat{\Sigma}_{\Gamma^2}$ ***exhibits measurement contextuality with*** $|\phi_3\rangle$.

ϕ	$\sum_x W_\phi(x)W_{\Sigma_{\Gamma^2}}(x)$	$\sum_x W_\phi(x)W^{\hbar^0}_{\Sigma_{\Gamma^2}}(x)$	$\sum_x W_\phi(x)W^{\hbar}_{\Sigma_{\Gamma^2}}(x)$
1	$5 - 2\sqrt{5}$	$\frac{1}{12}\left(25 - 9\sqrt{5}\right)$	$\frac{5}{12}\left(7 - 3\sqrt{5}\right) \approx 0.12$
2	$5 - 2\sqrt{5}$	$\frac{1}{12}\left(25 - 9\sqrt{5}\right)$	$\frac{5}{12}\left(7 - 3\sqrt{5}\right) \approx 0.12$
3	$5 - \sqrt{5}$	$\frac{1}{6}\left(5 - \sqrt{5}\right)$	$\frac{5}{6}\left(5 - \sqrt{5}\right) \approx$ **2.30**
$\vdots$	$\vdots$	$\vdots$	< 2

Table 2: $\hat{\Sigma}_{\Gamma^2}$ expectation value w.r.t. orders of $\hbar$ in WWM.

Future Directions

We are working on the development classical algorithms based on the WWM formalism for quantum simulation using contextuality as a resource.