Semiclassical Derivation of the Gottesman-Knill Theorem and Universal Quantum Computation for Qudits with Odd Dimension

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Abstract

In a restatement of the Gottesman-Knill theorem involving $\hbar$, we prove that, in discrete systems, computing within the stabilizer states can be fully accomplished by a propagator truncated to order $\hbar^0$. Universal quantum computing can be fully accomplished by a propagator truncated to order $\hbar^1$ and it is not necessary to include higher orders.

Motivation for this Study

Recently, the exploration of the power of "quantum" resources over "classical" resources in the field of quantum information has led to the development of **discrete** Wigner formulations. ***This progress begs for the extension of semiclassical propagator techniques from the continuous to discrete world.***

- semiclassical propagators traditionally used in the continuous infinite-dimensional context
- they can lend their physical intuition and perhaps resolve some outstanding problems in the field, like Clifford group operators (Gottesman-Knill Theorem)

Some Necessary and Useful Vocabulary

- a "dit" is a classical unit of information that can be equal to $0, 1, \ldots, d$
- a "qudit" is the quantum analogue of a "dit" and can be in a *superposition* of $0, 1, \ldots, d$
- in the quantum circuit model, we are always concerned *states* that propagate or evolve under *operations*
  - states Ψ are usually denoted as "ket" vectors: $|\Psi\rangle$
  - operators O are usually given "hats"—$\hat{O}$—and they are frequently shown as matrices since they are linear
  - a state Ψ that becomes Ψ' under the operator O is denoted: $\hat{O}|\Psi\rangle = |\Psi'\rangle$
- like any vector, we can choose to view $|\Psi\rangle$ in any basis we want
  - the above result is from the position basis: $\Psi(q)$
  - we can also use the "center" basis, a.k.a. Ψ's **Wigner function** $\Psi(p,q)$, and see it both momentum and position
- an operator $\hat{O}$ that takes $|q\rangle$ to $|q'\rangle$ can be expressed in terms of its Hamiltonian $H_{\hat{O}}$ in a propagator that sums over paths q_{tj} that can be expanded in powers of $\hbar$:

$$\left\langle q' \middle| e^{-\frac{i}{\hbar}H_{\hat{O}}t} \middle| q \right\rangle = \sum_j^{\text{cl. paths}} \int \mathcal{D}[q_{tj}] \exp\left[\frac{i}{\hbar}\left(\overbrace{S_{\hat{O}}[q_{tj}]}^{\mathcal{O}(\hbar^0)} + \overbrace{\delta S_{\hat{O}}[q_{tj}]}^{\mathcal{O}(\hbar^1)} + \overbrace{\frac{1}{2}\delta^2 S_{\hat{O}}[q_{tj}]}^{\mathcal{O}(\hbar^2)} + \ldots\right)\right], \quad (1)$$

- the $\mathcal{O}(\hbar^n)$ necessary to describe a particular operation on states can be used to define it
  - $\mathcal{O}(\hbar^0)$ = "classical" operation
  - $\mathcal{O}(\hbar^n)$ for $n > 0$ = operation has some "quantumness"

Background: Infinite vs Finite Dimensional World

- continuous-infinite dimensional system
  - position q and momentum p take on values on the real line $\mathbb{R}$
  - realm of continuous semiclassics
- discrete finite dimensional
  - position q and momentum p take on *discrete* values
  - realm of quantum circuit models

"Classical" Evolution in Continuous Systems

In continuous systems, a Gaussian state can be defined as:

$$\Psi_\beta(q) = \left[\pi^{-1}\left(\mathrm{Re}\,\Sigma_\beta\right)\right]^{\frac{1}{4}} \exp\left[\frac{i}{\hbar}p_\beta\left(q - q_\beta\right) - \frac{1}{2}\Sigma_\beta\left(q - q_\beta\right)^2\right]. \quad (2)$$

$q_\beta \in \mathbb{R}$ is the central position, $p_\beta \in \mathbb{R}$ is the central momentum, $\mathrm{Re}\,\Sigma_\beta$ is proportional to the spread of the Gaussian and $\mathrm{Im}\,\Sigma_\beta$ captures p-q correlation.

Classical Evolution in the Continuous World

Operations that take Gaussians to Gaussians can be fully described by a propagator truncated to order $\hbar^0$.

The Wigner function of a Gaussian is itself a two-dimensional Gaussian in phase space (p, q):

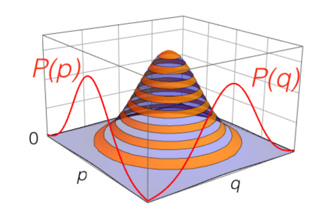

Figure: The Wigner function of a Gaussian state like Eq. 2. The projections of the Wigner function onto the position or momentum axes (shown in red) give the probability distributions for those quantities.

Discrete Phase Space

Before we can define a Wigner function in phase space, we must first define an adequate pair of conjugate degrees of freedom p and q for discrete systems.

By analogy with continuous finite translation operators, we re-express the shift $\hat{X}$ and boost $\hat{Z}$ operators in terms of conjugate $\hat{p}$ and $\hat{q}$ operators:

$$\hat{Z}|n\rangle = e^{\frac{2\pi i}{d}\hat{q}}|n\rangle = e^{\frac{2\pi i}{d}n}|n\rangle, \quad (3)$$

$$\hat{X}|n\rangle = e^{\frac{2\pi i}{d}\hat{p}}|n\rangle = |n \oplus 1\rangle, \quad (4)$$

where $\oplus$ denotes mod-d arithmetic. We note that $\hat{X}$ and $\hat{Z}$ are related to each other by discrete Fourier transform:

$$\hat{X} = \hat{F}^\dagger \hat{Z} \hat{F}. \quad (5)$$

Thus,

$$\hat{q} = \frac{d}{2\pi i}\log\hat{Z} = \sum_{n\in\{0,1,\ldots,d\}} n\,|n\rangle\langle n|, \quad (6)$$

and

$$\hat{p} = \hat{F}\hat{q}\hat{F}^\dagger. \quad (7)$$

With p and q now defined, we can define the discrete Wigner function of a pure state $|\Psi\rangle$:

$$\Psi_W(p,q) = (2\pi\hbar)^{-1} = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \Psi\left(q + \frac{q}{2}\right)\Psi^*\left(q - \frac{\xi}{2}\right)e^{-\frac{i}{\hbar}\xi p}.$$

Stabilizer States

If Ψ_β is a **stabilizer state**, then there exists a θ_β and an $\eta_\beta \in \{0, 1, \ldots, d\}$ such that its Wigner function is:

$$\Psi_{\beta_W}(p,q) = d^{-1} \sum_{\substack{\xi\in\\ \{0,1,\ldots,d\}}} \exp\left[\frac{2\pi i}{d}\xi\cdot\left(\eta_\beta - p + 2\theta_\beta q\right)\right]. \quad (8)$$

Therefore, one finds that the Wigner function is the discrete Fourier sum equal to $\delta_{\eta_\beta - p + 2\theta_\beta q}$. For $\theta_\beta = 0$ the state is a momentum state at p. Finite θ_β rotates that momentum state in phase space in "steps" such that it always lies along the discrete Weyl phase space points $(p, q) \in \{0, 1, \ldots, d\}$.

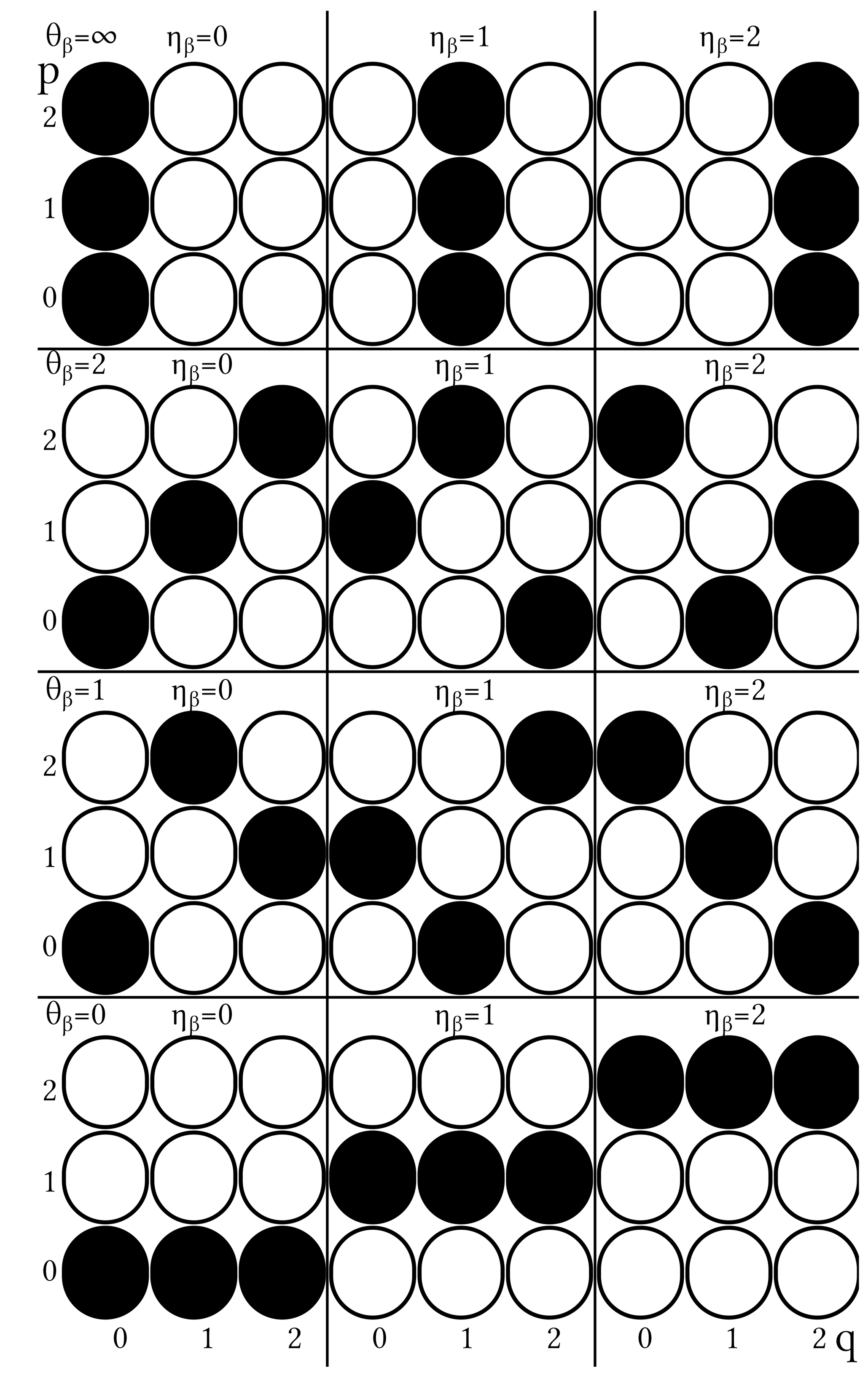

Figure: Table of qudit (where $d = 3$, a qutrit) stabilizer states for the different allowed values of θ_β and $\eta_\beta \in 0, 1, 2$.

Gottesman-Knill Theorem

The Clifford group are a set of quantum operators on qudits that can be simulated very efficiently. More precisely, by the Gottesman-Knill Theorem, for n qudits, a quantum circuit of a Clifford gate can be simulated using [2, 1]:

- $\mathcal{O}(n)$ operations on a classical computer
- measurements of outcomes require $\mathcal{O}(n^2)$ operations
- $\mathcal{O}(n^2)$ dits necessary to store in memory to describe the state

Discrete Propagator

With p and q now defined, we can also develop a path integral for the discrete case [3]:

$$U_t(x) = \left\langle \sum_j \left\{\det\left[1 + \frac{1}{2}\mathcal{J}\frac{\partial^2 S_{tj}}{\partial x^2}\right]\right\}^{\frac{1}{2}} e^{\frac{i}{\hbar}S_{tj}(x)}e^{i\theta_k}\right\rangle_k + \mathcal{O}(\hbar^2), \quad (9)$$

where an additional average must be taken over the center points k that are equivalent because of the periodic boundary conditions and accrue a phase θ_k. The derivative $\frac{\partial^2 S_{tj}}{\partial x^2}$ is performed over the continuous function S_{tj} but only evaluated at the discrete Weyl phase space points (p, q).

Classical Evolution in the Discrete World

Clifford operations that act on stabilizer states can be fully described by a propagator defined on the discrete phase space truncated to order $\hbar^0$.

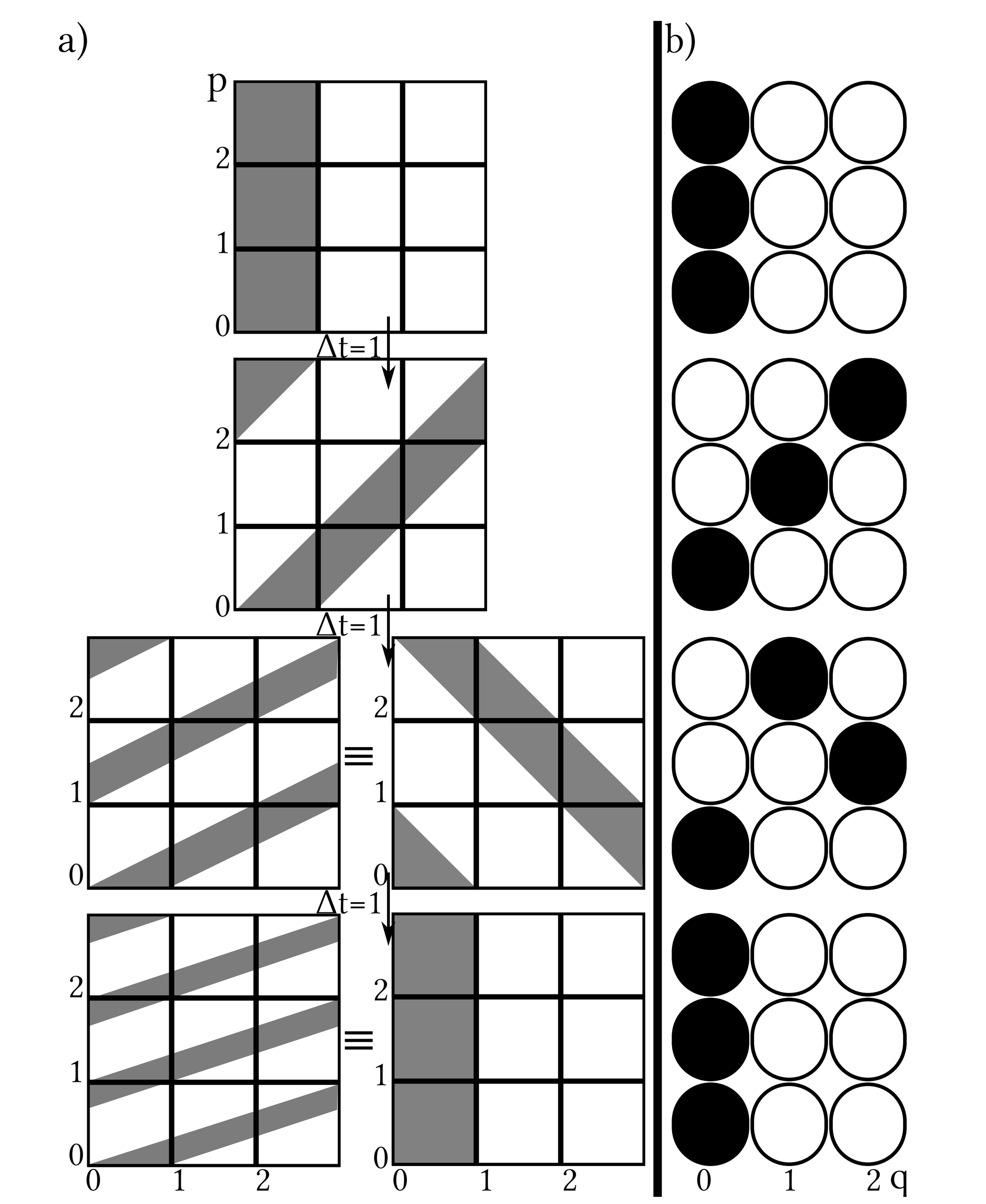

Figure: "Free" propagation of an initial stabilizer state in a) continuous and b) discrete phase space.

Conclusion

continuous	discrete
Gaussian	stabilizer state
harmonic Hamiltonian	—"— with integer coefficients
"classical" operations require single path integral contribution truncated at $\mathcal{O}(\hbar^0)$	—"— —"— —"—
"quantum" operations require **infinite** path integral contributions truncated at $\mathcal{O}(\hbar^1)$	—"— **finite** —"— —"—

Table: Continuous/Discrete Comparison

Future Direction

We hope to apply this result to develop quantum error-correction in adiabatic quantum computing.

References

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[3] A. M. Rivas, M. Saraceno, and A. O. De Almeida. Quantization of multidimensional cat maps. *Nonlinearity*, 13(2):341, 2000.