Finite Odd-Dimensional Wigner Formulation and the Stabilizer Subtheory

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I derive the finite odd-dimensional (discrete) Wigner-Weyl-Moyal formalism and show how stabilizer Wigner functions are the same as the tableau formalism for the stabilizer subtheory. This is a summary of *Kocia, Lucas, Yifei Huang, and Peter Love. "Discrete Wigner function derivation of the Aaronson-Gottesman tableau algorithm." Entropy 19.7 (2017): 353.*

I. BASICS

Let d be odd: $|j\rangle$ is the computational basis for $0 \leq j \leq (d-1)$. We define the "boost" operator

$$\hat{Z}\,|n\rangle = \omega^n\,|n\rangle\,, \tag{1}$$

and a discrete Fourier transform,

$$\hat{F} = \sqrt{\frac{\phi}{2\pi}} \sum_{m,n}^{d-1} \omega^{mn}\,|m\rangle\,\langle n| \tag{2}$$

for $\omega \in \mathbb{C}$ s.t. $|\omega| = 1$, i.e. $\omega = \exp i\phi$.

We can define a "shift" operator from these:

$$\hat{X}\,|n\rangle \equiv \hat{F}^\dagger \hat{Z} \hat{F}\,|n\rangle = |n \oplus 1\rangle\,, \tag{3}$$

where $\oplus$ denotes addition mod d.

This is easy to show:

$$\begin{aligned} \hat{F}^\dagger \hat{Z} \hat{F} &= \frac{\phi}{2\pi} \sum_{m,n,o} (\omega^{mn})^*\,|m\rangle\,\langle n|\,\omega^n\,|n\rangle\,\langle o|\,\omega^{no} \qquad (4)\\ &= \frac{\phi}{2\pi} \sum_{m,n,o} \omega^{n(-m+1+o)}\,|m\rangle\,\langle o| \\ &= \frac{\phi}{2\pi}\frac{2\pi}{\phi} \sum_{m,o} \delta_{m,o+1}\,|m\rangle\,\langle o| \\ &= \sum_m |o \oplus 1\rangle\,\langle o|\,, \end{aligned}$$

where $\delta i, j$ is a Kronecker delta function with arguments taken mod d.

This implies that

$$\hat{Z}\hat{X} = \omega \hat{X} \hat{Z}. \tag{5}$$

$$\hat{I} = (\hat{Z}\hat{X})^d = \hat{Z}^d \hat{X}^d = \omega^d \hat{X}^d \hat{Z}^d. \tag{6}$$

Therefore, $\omega^d = 1$.

We choose $\omega = \exp 2\pi i/d$.

Now we define

$$\hat{T}(\xi_p, \xi_q) = \alpha^{\xi_p \xi_q} \hat{Z}^{\xi_p} \hat{X}^{\xi_q}. \tag{7}$$

We want this to act like a translation operator and so

$$\hat{T}^{-1}(\xi_p, \xi_q) = \hat{T}(-\xi_p, -\xi_q). \tag{8}$$

This implies that

$$\alpha^{\xi_p \xi_q} \hat{X}^{-\xi_q} \hat{Z}^{-\xi_p} = \alpha^{\xi_p \xi_q} \omega. \tag{9}$$

$$\begin{aligned} 1 &= \hat{T}(-\xi_p, -\xi_q)\hat{T}(\xi_p, \xi_q) \qquad (10)\\ &= \alpha^{2\xi_p\xi_q} \hat{Z}^{-\xi_p} \hat{X}^{-\xi_q} \hat{Z}^{\xi_p} \hat{X}^{\xi_q} \\ &= \alpha^{2\xi_p\xi_q} \omega^{-\xi_p\xi_q} \hat{Z}^{-\xi_p} \hat{Z}^{\xi_p} \hat{X}^{-\xi_q} \hat{X}^{\xi_q} \\ &= \alpha^{2\xi_p\xi_q} \omega^{-\xi_q\xi_p} \\ &= (\alpha^2 \omega^{-1})^{\xi_p\xi_q}. \qquad (11) \end{aligned}$$

Therefore, $\alpha = \omega^{-1/2}$.

In odd d arithmetic, $-1/2 \equiv (d+1)/2$.

Therefore,

$$\hat{T}(\xi_p, \xi_q) = \omega^{-d+1)/2} \hat{Z}^{\xi_p} \hat{X}_{\xi_q} \tag{12}$$

and

$$\hat{T}^{-1}(\xi_p, \xi_q) = \hat{T}(-\xi_p, -\xi_q) = \hat{T}^\dagger(\xi_p, \xi_q). \tag{13}$$

$\hat{T}$ are Hilbert-Schmidt orthogonal and so can be used as a complete operator basis for any operator $\hat{A}$:

$$\begin{aligned} \hat{A} &= d^{-1} \sum_{\xi_p, \xi_q \in \mathbb{Z}/d\mathbb{Z}} \mathrm{Tr}(\hat{T}(-\xi_p, -\xi_q)\hat{A})\hat{T}(\xi_p, \xi_q) \qquad (14)\\ &\equiv d^{-1} \sum_{\xi_p, \xi_q \in \mathbb{Z}/d\mathbb{Z}} A_\xi(\xi_p, \xi_q)\hat{T}(\xi_p, \xi_q). \end{aligned}$$

We also note that the T satisfy the translation group structure with an additional phase:

$$\hat{T}(\boldsymbol{\xi}_2)\hat{T}(\boldsymbol{\xi}_1) = \hat{T}(\xi_1 + \xi_2)\omega^{(-\xi_{1p}\xi_{2q} + \xi_{1q}\xi_{2p})(d+1)/2}. \tag{15}$$

We define a *symplectic* Fourier transform of $\hat{T}$:

$$\hat{R}(x_p, x_q) = d^{-1} \sum_{\xi_p, \xi_q \in \mathbb{Z}/d\mathbb{Z}} \omega^{\xi_p x_q - \xi_q x_p} \hat{T}(\xi_p, \xi_q). \tag{16}$$

It is easy to show that $\hat{R}$ satisfies the following properties of a reflection operator, with added phase:

$$\hat{R}(\boldsymbol{x})\hat{T}(\boldsymbol{\xi}) = \hat{R}(\boldsymbol{x} - \boldsymbol{\xi}/2)\omega^{x_p\xi_q - x_q\xi_p}, \tag{17}$$

$$\hat{T}(\boldsymbol{\xi})\hat{R}(\boldsymbol{x}) = \hat{R}(\boldsymbol{x} + \boldsymbol{\xi}/2)\omega^{-x_p\xi_q + x_q\xi_p}, \tag{18}$$

$$\hat{R}(\boldsymbol{x}_1)\hat{R}(\boldsymbol{x}_2) = \hat{T}(2(\boldsymbol{x}_2 - \boldsymbol{x}_1))\omega^{x_{1p}x_{2q} - x_{1q}x_{2p}}. \tag{19}$$

This implies that $\hat{R}^2 = \hat{I}$.

Therefore, $\hat{R}$ are Hilbert-Schmidt orthogonal, Hermitian, self-inverse and unitary operators. This means that they can serve as an operator basis for any operator $\hat{A}$ with coefficients (defined $A_x(x_p, x_q)$ below) which will be real-valued:

$$\begin{aligned}\hat{A} &= d^{-1} \sum_{x_p, x_q} \mathrm{Tr}(\hat{R}(x_p, x_q)\hat{A})\hat{R}(x_p, x_q) \\ &\equiv A_x(x_p, x_q)\hat{R}(x_p, x_q).\end{aligned} \tag{20}$$

It is easy to show that $A_x(x_p, x_q)$ satisfies the following properties:

$$\sum_{x_p, x_q \in \mathbb{Z}/d\mathbb{Z}} A_x(x_p, x_q) = 1, \tag{21}$$

$$\sum_{x_p \in \mathbb{Z}/d\mathbb{Z}} A_x(x_p, x_q) = \left\langle x_q \middle| \hat{A} \middle| x_q \right\rangle, \tag{22}$$

$$\sum_{x_q \in \mathbb{Z}/d\mathbb{Z}} A_x(x_p, x_q) = \left\langle x_p \middle| \hat{A} \middle| x_p \right\rangle. \tag{23}$$

When $\hat{A}$ is an operator, we will call $A_x(x_p, x_q)$ the Weyl operator. When $\hat{A} = \hat{\rho}$ is a state, we will call $\rho_x(x_p, x_q)$ a Wigner function.

This representation of finite odd-dimensional quantum states is especially simple for the Clifford subtheory. For stabilizer states $\hat{\rho}$, Gross [1] proved

$$\rho_x(x_p, x_q) \in \mathbb{R}^+ \cup \{0\}. \tag{24}$$

For Clifford gates $\hat{O}$, Almeida [2] proved

$$O_x(x_p, x_q) = \exp 2\pi i S(x_p, x_q)/d, \tag{25}$$

where $S(x_p, x_q)$ is a *quadratic* function with *integer* coefficients. It can be shown that this means that it transforms Wigner functions (of all states, not just stabilizer states)

$$(\hat{O}^\dagger \hat{\rho} \hat{O})(x_p, x_q) = \rho_x \left(\mathcal{M}_{\hat{O}} \begin{pmatrix} x_p \\ x_q \end{pmatrix} \oplus \begin{pmatrix} \alpha_p \\ \alpha_q \end{pmatrix} \right), \tag{26}$$

for a symplectic matrix $\mathcal{M} \in (\mathbb{Z}/d\mathbb{Z})^{2\times 2}$ and $\boldsymbol{\alpha} \in (\mathbb{Z}/d\mathbb{Z})^2$.

In other words, Clifford gates rearrange the indices of Wigner functions in a manner that preserves their symplectic area... To be continued.

[1] David Gross. Hudson's theorem for finite-dimensional quantum systems. *Journal of mathematical physics*, 47(12):122107, 2006.

[2] AMF Rivas and AM Ozorio De Almeida. The weyl representation on the torus. *Annals of Physics*, 276(2):223–256, 1999.